**Using a Discrete Fourier Series to Model Data**

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| In the previous sections we have used a Fourier series to approximate a function over an interval. Fourier series also can be used to model data on an interval. A Fourier series that is based on discrete data, rather than on a piecewise continuous function, is called a ***discrete Fourier series.*** A continuous function is sampled to reduce the [continuous signal](http://en.wikipedia.org/wiki/Continuous_signal) to a [discrete signal](http://en.wikipedia.org/wiki/Discrete_signal). A common example is the conversion of a [sound wave](http://en.wikipedia.org/wiki/Sound_wave) (a continuous signal) to a sequence of data points. | http://upload.wikimedia.org/wikipedia/commons/thumb/5/50/Signal_Sampling.png/300px-Signal_Sampling.png  The continuous signal (green colored line) is converted into the discrete samples (blue vertical lines).  From <http://en.wikipedia.org/wiki/Sampling_(signal_processing)> |

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| *Example* Consider the set of data that consists of ten equally-spaced *x* and *y* values of the function  on the interval [*−π,π*]*.* If we divide the interval [−*π,π*]into nine subintervals, each of length we generate the ten *x* values that are paired with *y* values such that for  These values are shown rounded to hundredths and graphed below. Write a discrete Fourier series of order 4 to model these data. Note that the data in the table below have been rounded to 2 decimal places. The Fourier coefficients have been calculated using more decimal places.   |  |  |  |  |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | | ***x*** | −3.14 | −2.44 | −1.75 | −1.05 | −0.35 | 0.35 | 1.05 | 1.75 | 2.44 | 3.14 | | ***y*** | −8.87 | −4.97 | −2.05 | −0.10 | 0.88 | 0.88 | −0.10 | −2.05 | −4.97 | −8.87 |   Ten data points from |

*Solution* We need to find a way to create this Fourier series based only on the data from our sample, without knowing the function *f.* Recall that when we know the equation of the function *f*, the Fourier coefficients are determined as follows:

, , and .

The Fourier series based on these coefficients is

Since we are assuming that we do not know *f,* we will use these ten data values to generate the required values of *ak* and *bk* . We accomplish this by rewriting definite integrals as summations. The definite integral used to define *,* can be approximated by the left Riemann sum . The ten data points divide the interval [−π, π] into nine equal subintervals. Since the interval length is 2π we know that .We can compute *a*0with the sum . Rounded to hundredths, this sum is −2.37. Therefore, we conclude that *a*0= −2.37 is a reasonable approximation for the constant term in the discrete Fourier series.

The value of ** in a Fourier series is defined by which can be approximated with the Riemann sum .. Again, so the value of *a1* in the discrete Fourier series is

Evaluating this sum and rounding to hundredths, we find that *a1* = 4.17. Continuing in a similar fashion with higher order terms, we find that *a2* = −1.18, *a3* = 0.65, and *a4* = −0.50.

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| The values of the coefficients *bk* in a discrete Fourier series are determined in a similar way by replacing    with  ,  where *N* is the number of data points. In this example, the computed values of *b1, b2, b3,* and *b4* are all less than 10-10 (this is not a surprise, since the data clearly come from an even | Graphs of ten data points and *F4* (*x*) |

function). These values are relatively small compared to the *aj*'s*,* so we can leave the corresponding terms out of the series without losing significant accuracy. Therefore, the discrete Fourier approximation series of order 4 based on the ten data points is

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In general, if *N* is the number of data points we have on the interval [ −π, π] , then the spacing between data points is . When we calculate the value of any *ak* or *bk* in a discrete Fourier series, we use the ordered pairs (*xi , yi*)in the data set we are modeling rather than the values of a function as in previous sections. The coefficient *ao* in the discrete Fourier series is defined by

In general, the coefficients  in the discrete Fourier series are defined by

Similarly, the coefficients ** in the discrete Fourier series are defined by

To summarize, we know that the discrete Fourier series for a data set containing *N* points of the form (*xi , yi*) on [−π, π] is defined by

where

, , and .

If the interval of interest is [*c,* *d*]rather than [ −π, π] , we make the same type of transformations as in the continuous case.

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| The discrete Fourier series for a data set containing *N* points (*xi , yi*)on [*c, d*]is defined by  where β =  and  ,  ,  and  . |

1. Our work in the example in this section showed that the discrete Fourier series of order 4 based on 10 data points from on [ -π, π] is

We know from our previous work that the Fourier series of order 4 for on[ -π, π]  *is*

a. Calculate the relative error in the coefficients of successive terms. That is, by what percent does each coefficient in the discrete Fourier series differ from its counterpart in the original Fourier series?

b. Create a set of twenty equally-spaced data points from the function *f(x)* = 1- *x2* on the interval

[ −π, π]. Determine the coefficients in the discrete Fourier series of order 4 for this data set. Repeat this procedure for a data set of forty equally-spaced points. How do these coefficients compare to the coefficients in the series created from the function itself?